Journal of Engineering Mathematics 20 (1986) 189–201

A cartesian-tensor solution of the viscous creeping-motion equations

R. NIEFER * and P.N. KALONI

Department of Mathematics, University of Windsor, Windsor, Ontario, Canada

(Received January 22, 1986)

Summary.

A solution to the viscous creeping-motion equations is developed, using cartesian tensors. The solution is motivated by the form of the boundary conditions of the problem and is, therefore, easily applicable to situations where the boundary conditions are expressible in cartesian-tensor form. Several illustrations of the method are given to show how the solutions for the pressure and velocity components can be constructed, directly and easily, from the general solution.

1. Introduction

In studying the rheological properties of suspensions, colloidal systems and polymer composites we often need solutions of the viscous creeping-motion equations. If we limit the discussion to steady, incompressible motion of such fluids, then it turns out that the equations determining the velocity vector and the pressure become linear partial differential equations. As a result, a good variety of solutions of such equations applicable to different dimensions is available. Basically these solutions fall into three main categories, as has been summarized by Happel and Brenner [1]. The first method is that of Lamb [2], which uses spherical harmonics and which involves Legendre and associated Legendre polynomials. This method, though quite general in itself, becomes quite cumbersome in certain cases, because of its extreme generality. The second method uses a stream-function technique which can be used only for two-dimensional problems and for those three-dimensional problems that exhibit some type of axial symmetry. Finally, we have the singularity method, first originated by Lorentz [3], which, though easy to apply, requires much guesswork to determine the appropriate singularities necessary to solve a particular problem.

Most of the fluid-flow problems commonly encountered can be solved by at least one of the above methods. In many instances, however, the boundary conditions are presented in a fashion that is not readily translated to a form that is directly applicable to any of these methods. It is well-known that certain types of auxiliary conditions are appropriate only to certain corresponding types of partial differential equations. For example,

* *Present address*: Faculty of Engineering and Applied Science, Memorial University of Newfoundland, St. John's, Newfoundland, Canada.

problems of potential theory lead to Dirichlet's problem and are, therefore, appropriate for elliptic equations. In the literature, sometimes the boundary conditions are given in a cartesian-tensor from which cannot always be easily put into a form that is usable for any of the methods mentioned above.

The purpose of this paper is to develop, partially, another form for the general solution of the viscous creeping-motion equations, based on the use of cartesian tensors. The solution thus generated will be directly applicable to problems for which the boundary conditions are given in cartesian-tensor form and to those that can easily be written in this manner. In the present work, use will be made of arbitrary, spatially constant, second- and third-order tensors to generate the solution. The solution so obtained will be applied to a number of problems to illustrate its simplicity and usefulness. Even though the problems considered at present involve a particle of spherical shape, further applications of the method, to take into account other shapes, will be considered in a future paper.

2. Development of the solution

We consider the flow field satisfying the creeping-motion equation, or Stokes equation,

$$\mu\nabla^2\mathbf{u} = \nabla\bar{p}, \tag{1}$$

and the continuity equation

$$\nabla\cdot\mathbf{u} = 0. \tag{2}$$

Here $\mathbf{u}$ is the velocity field, $\bar{p}$ the pressure and μ the dynamic viscosity. We assume that $\bar{p}$ contains any conservative extraneous body forces that may be present in the system. It is well known that solving the above system is equivalent to finding solutions of the equations

$$\nabla^2 p = 0, \tag{3}$$

$$\nabla^2\mathbf{u} = \nabla p, \tag{4}$$

where $p = \mu^{-1}\bar{p}$, and subject to the constraint that $\mathbf{u}$, once determined, must be restricted so that the continuity equation (2) is satisfied.

In order to generate solutions which will involve spatially constant second- and third-order tensors a_{ij} and b_{ijk}, respectively, it is necessary to determine the scalar invariants involving these tensors in combination with the position vector $\mathbf{r}$. Such scalar invariants linear in a_{ij} and b_{ijk} are:

$$a_{ii},\quad \epsilon_{iju}a_{uj}x_i,\quad a_{ij}x_ix_j,\quad b_{imm}x_i,\quad b_{mim}x_i,\quad b_{mmi}x_i,$$

$$b_{ijk}x_ix_jx_k. \tag{5}$$

Hence a general form for $p(\mathbf{r})$ is assumed to be

$$p(\mathbf{r}) = H^0(r)a_{ii} + H^1(r)\epsilon_{ijk}a_{kj}x_i + H^2(r)b_{imm}x_i + H^3(r)b_{mim}x_i + H^4(r)b_{mmi}x_i + H^5(r)a_{ij}x_ix_j + H^6(r)b_{ijk}x_ix_jx_k \tag{6}$$

where $r = (x_ix_i)^{1/2}$.

Since a_{ij} and b_{ijk} are general tensors, the function $H^p(r)$, in order to satisfy (3), must satisfy an equation of the form

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}H^p(r) + \frac{n+2m-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}H^p(r) = G(r) \tag{7}$$

where n is the dimension of the Euclidean space, m is the order of the coefficient tensor, and $G(r)$ is either a known function or zero. We shall here given solutions for $n = 3$ and at the end mention will be made of the special changes necessary for the case $n = 2$. Thus, for the function $p(\boldsymbol{r})$ it is found that

$$\begin{aligned}
H^0(r) &= -\frac{1}{n}A_1^5r^{-n} + A_1^0r^{-(n-2)} + A_2^0 - \frac{1}{n}A_2^5r^2,\\
H^1(r) &= A_1^1r^{-n} + A_2^1,\\
H^2(r) &= -\frac{1}{n+2}A_1^6r^{-(n+2)} + A_1^2r^{-n} + A_2^2 + \frac{1}{n+2}A_2^6r,\\
H^3(r) &= -\frac{1}{n+2}A_1^6r^{-(n+2)} + A_1^3r^{-n} + A_2^3 - \frac{1}{n+2}A_2^6r^2,\\
H^4(r) &= -\frac{1}{n+2}A_1^6r^{-(n+2)} + A_1^4r^{-n} + A_2^4 - \frac{1}{n+2}A_2^6r^2,\\
H^5(r) &= A_1^5r^{-(n+2)} + A_2^5,\\
H^6(r) &= A_1^6r^{-(n+4)} + A_2^6,
\end{aligned} \tag{8}$$

where A_j^i are arbitrary constants.

From equations (4) and (6) the general form for the velocity component in the direction x_p is given by

$$\begin{aligned}
u_p(\mathbf{r}) &= h_1^0(r)a_{ii}x_p + h_1^1(r)\epsilon_{ijk}a_{kj}x_ix_p + h_2^1(r)\epsilon_{pjk}a_{kj}\\
&+ h_1^2(r)b_{imm}x_ix_p + h_2^2(r)b_{pmm} + h_1^3(r)b_{mim}x_ix_p\\
&+ h_2^3(r)b_{mpm} + h_1^4(r)b_{mmi}x_ix_p + h_2^4(r)b_{mmp}\\
&+ h_1^5(r)a_{ij}x_ix_jx_p + h_2^5(r)a_{pj}x_j + h_3^5(r)a_{jp}x_j\\
&+ h_1^6(r)b_{ijk}x_ix_jx_kx_p + h_2^6(r)b_{pjk}x_jx_k + h_3^6(r)b_{jpk}x_jx_k\\
&+ h_4^6(r)b_{jkp}x_jx_k.
\end{aligned} \tag{9}$$

Substitution of (9) and (6) in (4) requires that the functions $h_q^p(r)$ satisfy the following differential equations:

$$\frac{d^2}{dr^2}h_1^p(r)+\frac{n+2m+1}{r}\frac{d}{dr}h_i^p(r)=g_1(r), \tag{10}$$

where $p\in\{0,1,2\ldots6\}$ and $q=1$, and

$$\frac{d^2}{dr^2}h_q^p+\frac{n+2m-3}{r}\frac{d}{dr}h_q^p(r)=g_2(r), \tag{11}$$

where $q\neq1$, $p\in\{0,1,2\ldots6\}$. In the above, n and m are again the dimensions of the space and orders of the coefficient tensors a_{ij} and b_{ijk}, respectively, and $g_1(r)$, $g_2(r)$ are either known functions or zero. The solutions for equations (10) and (11) are given by

$$h_1^0(r)=-\frac{1}{n+2}A_3^5r^{-(n+2)}+A_3^0r^{-n}+\tfrac{1}{2}A_1^0r^{2-n}+A_4^0-\frac{(A_2^5+nA_4^5)r^2}{n(n+2)},$$

$$h_1^1(r)=A_3^1r^{-(n+2)}+\tfrac{1}{2}A_1^1r^{-n}+A_4^1,$$

$$h_2^1(r)=-\frac{1}{n}A_3^1r^{-n}+A_5^1r^{2-n}+A_6^1+\frac{A_2^1-2A_4^1}{2n}r^2,$$

$$h_1^2(r)=-\frac{1}{n+4}A_3^6r^{-(n+4)}+A_3^2r^{-(n+2)}+\tfrac{1}{2}A_1^2r^{-n}+A_4^2-\frac{A_2^6+(n+2)A_4^6}{(n+2)(n+4)}r^2,$$

$$h_2^2(r)=\frac{1}{(n+2)(n+4)}A_3^6r^{-(n+2)}-\frac{A_1^6+2(n+2)(A_3^2+A_5^6)}{2n(n+2)}r^{-n}$$
$$+A_5^2r^{2-n}+A_6^2+\frac{A_2^2-2(A_4^2+A_6^6)}{2n}r^2-\frac{A_2^6-2A_4^6}{2(n+2)(n+4)}r^4,$$

$$h_1^3(r)=\frac{1}{n+4}A_3^6r^{-(n+4)}+A_3^3r^{-(n+2)}+\tfrac{1}{2}A_1^3r^{-n}+A_4^3-\frac{A_2^6+(n+2)A_4^6}{(n+2)(n+4)}r^2,$$

$$h_2^3(r)=\frac{1}{(n+2)(n+4)}A_3^6r^{-(n+2)}-\frac{A_1^6+2(n+2)(A_3^3+A_7^6)}{2n(n+2)}r^{-n}$$
$$+A_5^3r^{2-n}+A_6^3+\frac{A_2^3-2(A_4^3+A_6^6)}{2n}r^2-\frac{A_2^6-2A_4^6}{2(n+2)(n+4)}r^4,$$

$$h_1^4(r)=\frac{1}{n+4}A_3^6r^{-(n+4)}+A_3^4r^{-(n+2)}+\tfrac{1}{2}A_1^4r^{-n}+A_4^4-\frac{A_2^6+(n+2)A_4^6}{(n+2)(n+4)}r^2,$$

$$h_2^4(r)=\frac{1}{(n+2)(n+4)}A_3^6r^{-(n+2)}-\frac{A_1^6+2(n+2)(A_3^4+A_9^6)}{2n(n+2)}r^{-n}$$

$$+A_5^4r^{2-n}+A_6^4+\frac{A_2^4-2(A_4^4+A_{10}^6)}{2n}r^2-\frac{A_2^6-2A_4^6}{2(n+2)(n+4)}r^4,$$

$$h_1^5(r)=A_3^5r^{-(n+4)}+\tfrac{1}{2}A_1^5r^{-(n+2)}+A_4^5,$$

$$h_2^5(r)=-\frac{1}{n+2}A_3^5r^{-(n+2)}+A_5^5r^{-n}+A_6^5+\frac{A_2^5-2A_4^5}{2(n+2)}r^2,$$

$$h_3^5(r)=-\frac{-1}{n+2}A_3^5r^{-(n+2)}+A_7^5r^{-n}+A_8^5+\frac{A_2^5-2A_4^5r}{2(n+2)},$$

$$h_1^6(r)=A_3^6r^{-(n+6)}=\tfrac{1}{2}A_1^6r^{-(n+4)}+A_4^6,$$

$$h_2^6(r)=\frac{-1}{n+4}A_3^6r^{-(n+4)}+A_5^6r^{-(n+2)}+A_6^6+\frac{A_2^6-2A_4^6}{2(n+4)}r^2,$$

$$h_3^6(r)=\frac{-1}{n+4}A_3^6r^{-(n+4)}+A_7^6r^{-(n+2)}+A_8^6+\frac{A_2^6-2A_4^6}{2(n+4)}r^2,$$

$$h_4^6(r)=\frac{-1}{n+4}A_3^6r^{-n(n+4)}+A_9^6r^{-(n+2)}+A_{10}^6+\frac{A_2^6-2A_4^6}{2(n+4)}r^2. \tag{12}$$

Substitution of (9), using (12), in (2) gives rise to a set of differential equations of the form

$$r\frac{\mathrm{d}}{\mathrm{d}r}h_1^p(r)+(n+m)h_1^p(r)=g_3(r) \tag{13}$$

where n, m are as before and $g_3(r)$ involves $h_q^p(r)$ for $q\neq 1$ and their derivatives. This set of equations when solved impose the following restrictions on the A_j^m:

$$A_1^6=-2(n+2)(A_5^6+A_7^6+A_9^6),$$

$$3A_2^6=-(n+1)(n+6)A_4^6,$$

$$2A_2^5=-n(n+4)A_4^5,$$

$$A_5^5=-A_7^5,$$

$$A_1^4=-2(2-n)A_5^4,$$

$$A_2^4=-(n-1)(n+2)A_4^4+2A_{10}^6-n(A_6^6+A_8^6),$$

$$A_1^3 = -2(2-n)A_5^3,$$

$$A_2^3 = -(n-1)(n+2)A_4^3 + 2A_8^6 - n\left(A_{10}^6 + A_6^6\right),$$

$$A_1^2 = -2(2-n)A_5^2,$$

$$A_2^2 = -(n-1)(n+2)A_4^2 + 2A_6^6 - n\left(A_8^6 + A_{10}^6\right),$$

$$A_1^1 = -2(2-n)A_5^1,$$

$$A_2^1 = -(n-1)(n+2)A_4^1,$$

$$A_1^0 = 0,$$

$$nA_4^0 = -\left(A_6^5 + A_8^5\right). \tag{14}$$

When the changes indicated by (14) have been made in (8) and (12), the functions $H^p(r)$, $h_q^p(r)$ for $n = 3$ are as follows:

$$H^0(r) = -\frac{1}{n}A_1^5 r^{-n} + A_2^0 - \frac{1}{n}A_2^5 r^2,$$

$$H^1(r) = A_1^1 r^{-n} + A_2^1,$$

$$H^2(r) = 2\left(A_5^6 + A_7^6 + A_9^6\right)r^{-(n+2)} + A_1^2 r^{-n} + A_2^2 - \frac{1}{n+2}A_2^6 r^2,$$

$$H^3(r) = 2\left(A_5^6 + A_7^6 + A_9^6\right)r^{-(n+2)} + A_1^3 r^{-n} + A_2^3 - \frac{1}{n+2}A_2^6 r^2,$$

$$H^4(r) = 2\left(A_5^6 + A_7^6 + A_9^6\right)r^{-(n+2)} + A_1^4 r^{-n} + A_2^4 - \frac{1}{n+2}A_2^6 r^2,$$

$$H^5(r) = A_1^5 r^{-(n+2)} + A_2^5,$$

$$H^6(r) = -2(n+2)\left(A_5^6 + A_7^6 + A_9^6\right)r^{-(n+4)} + A_2^6,$$

$$h_1^0(r) = \frac{-1}{n+2}A_3^5 r^{-(n+2)} + A_3^0 r^{-n} - \frac{A_6^5 + A_8^5}{n} - \frac{1}{n(n+4)}A_2^5 r^2,$$

$$h_1^1(r) = A_3^1 r^{-(n+2)} + \tfrac{1}{2}A_1^1 r^{-n} - \frac{1}{(n-1)(n+2)}A_2^1,$$

$$h_2^1(r) = -\frac{1}{n}A_3^1 r^{-n} + \frac{1}{2(n-2)}A_1^1 r^{2-n} + A_6^1 + \frac{n+1}{2(n-1)(n+2)}A_2^1 r^2,$$

$$h_1^2(r) = \frac{-1}{n+4}A_3^6 r^{-(n+4)} + A_3^2 r^{-(n+2)} + \tfrac{1}{2}A_1^2 r^{-n}$$
$$- \frac{A_2^2 - 2A_6^6 + n\left(A_8^6 + A_{10}^6\right)}{(n-1)(n+2)} - \frac{nA_1^6}{(n+1)(n+2)(n+6)}r^2 \tag{15}$$

$$h_2^2(r) = \frac{1}{(n+2)(n+4)} A_3^6 r^{-(n+2)} + \frac{1}{n}\left(A_7^6 + A_9^6 + A_3^2\right) r^{-n} + \frac{1}{2(n-2)} A_1^2 r^{2-n} + A_6^2$$

$$+ \frac{(n+1)\left(A_2^2 - 2A_6^6\right) + 2\left(A_6^6 + A_{10}^6\right)}{2(n-1)(n+2)} r^2 - \frac{n+3}{2(n+1)(n+2)(n+6)} A_2^6 r^4,$$

$$h_1^3(r) = \frac{-1}{n+4} A_3^6 r^{-(n+4)} + A_3^3 r^{-(n+2)} + \tfrac{1}{2} A_1^3 r^{-n} - \frac{A_2^3 - 2A_8^6 + n\left(A_6^6 + A_{10}^6\right)}{(n-1)(n+2)}$$

$$- \frac{nA_2^6}{(n+1)(n+2)(n+6)} r^2,$$

$$h_2^3(r) = \frac{1}{(n+2)(n+4)} A_3^6 r^{-(n+2)} + \frac{1}{n}\left(A_5^6 + A_9^6 - A_3^3\right) r^{-n} + \frac{1}{2(n-2)} A_1^3 r^{2-n} + A_6^3$$

$$+ \frac{(n+1)\left(A_2^3 - 2A_8^6\right) + 2\left(A_6^6 + A_{10}^6\right)}{2(n-1)(n+2)} r^2 - \frac{n+3}{2(n+1)(n+2)(n+6)} A_2^6 r^4,$$

$$h_1^4(r) = \frac{-1}{n+4} A_3^6 r^{-(n+4)} + A_3^4 r^{-(n+2)} + \tfrac{1}{2} A_1^4 r^{-n} - \frac{A_2^4 - 2A_{10}^6 + n\left(A_6^6 + A_8^6\right)}{(n-1)(n+2)}$$

$$- \frac{nA_2^6}{(n+1)(n+2)(n+6)} r^2,$$

$$h_2^4(r) = \frac{1}{(n+2)(n+4)} A_3^6 r^{-(n+2)} + \frac{1}{n}\left(A_5^6 + A_7^6 - A_3^4\right) r^{-n}$$

$$+ \frac{1}{2(n-2)} A_1^4 r^{2-n} + A_6^4 + \frac{(n+1)\left(A_2^4 - 2A_{10}^6\right) + 2\left(A_6^6 + A_8^6\right)}{2(n-1)(n+2)} r^2$$

$$- \frac{n+3}{2(n+1)(n+2)(n+6)} A_2^6 r^4,$$

$$h_1^5(r) = A_3^5 r^{-(n+4)} + \tfrac{1}{2} A_1^5 r^{-(n+2)} - \frac{2}{n(n+4)} A_2^5,$$

$$h_2^5(r) = \frac{-1}{n+2} A_3^5 r^{-(n+2)} + A_5^5 r^{-n} + A_6^5 + \frac{n+2}{2n(n+4)} A_2^5 r^2,$$

$$h_3^5(r) = \frac{-1}{n+2} A_3^5 r^{-(n+2)} - A_5^5 r^{-n} + A_8^5 + \frac{n+2}{2n(n+4)} A_2^5 r^2,$$

$$h_1^6(r) = A_3^6 r^{-(n+6)} - (n+2)\left(A_5^6 + A_7^6 + A_9^6\right) r^{-(n+4)} - \frac{3}{(n+1)(n+6)} A_2^6,$$

$$h_2^6(r)=\frac{-1}{n+4}A_3^6r^{-(n+4)}+A_5^6r^{-(n+2)}+A_6^6+\frac{n+3}{2(n+1)(n+6)}A_2^6r,$$

$$h_3^6(r)=\frac{-1}{n+4}A_3^6r^{-(n+4)}+A_7^6r^{-(n+2)}+A_8^6+\frac{n+3}{2(n+1)(n+6)}A_2^6r,$$

$$h_4^6(r)=\frac{-1}{n+4}A_3^6r^{-(n+4)}+A_9^6r^{-(n+2)}+A_{10}^6+\frac{n+3}{2(n+1)(n+6)}A_2^6r^2. \tag{15}$$

The functions $H^p(r)$ and $h_q^p(r)$, as given in equations (15), are valid in three-dimensional Euclidean space. However, for the special case of $n=2$, the functions presented above are still valid, provided that certain minor changes are made. The necessary changes in equations (15) are

$$r^{2-n}\to\ln r,\quad \frac{1}{2(n-2)}A_1^s\to\tfrac{1}{2}A_1^s$$

where $s\in\{1, 2, 3, 4\}$.

Equations (6) and (9) together with (15) constitute the general solution of the Stokes equations (1) and (2). Once the boundary conditions for a specific problem are prescribed, the corresponding general solution is easily generated from these equations. We remark that these solutions are not complete to the same degree as those of Lamb [2]. In the next section we shall illustrate their usefulness.

3. Illustrations

In the present section we consider some applications of the method.

(i) We first consider the Stokes flow for a uniform free stream past a solid sphere. With no loss of generality the boundary conditions can be written as

$$p_\infty=p_0,\quad \mathbf{u}_\infty=U\mathbf{e}_3,\quad \mathbf{u}=\mathbf{0}\qquad \text{on}\quad r=1. \tag{16}$$

The cartesian-tensor forms for these boundary conditions are

$$p_\infty=a_{ii},\quad u_{l\infty}=\epsilon_{ljk}a_{kj},\quad u_l=0\qquad \text{on}\quad r=1, \tag{17}$$

where we have assumed

$$a_{ii}=p_0,\quad a_{21}=U,\quad a_{ij}=0\quad \text{otherwise},$$

$$b_{ijk}=0\qquad \text{for all}\quad i,\ j,\ k. \tag{18}$$

From (17), it appears that the most natural choice for the coefficients in (15) is

$$A_n^m=0\qquad \text{except when}\quad m=0, 1. \tag{19}$$

Substitution of (15), using (19), in (6) and (9) gives

$$p = A_2^0 a_{ii} + \left(A_1^1 r^{-3} + A_2^1\right)\epsilon_{ijk} a_{kj} x_i,$$

$$u_l = A_3^0 r^{-3} a_{ii} x_l + \left(A_3^1 r^{-5} + \tfrac{1}{2} A_1^1 r^{-3} - \tfrac{1}{10} A_2^1\right)\epsilon_{ijk} a_{kj} x_i x_l$$

$$+ \left(-\tfrac{1}{3} A_3^1 r^{-3} + \tfrac{1}{2} A_1^1 r^{-1} + A_6^1 + \tfrac{1}{5} A_2^1 r^2\right)\epsilon_{ljk} a_{kj}. \tag{20}$$

In order for (20) to agree with the boundary conditions (17) it is necessary to choose

$$A_2^0 = 1, \quad A_3^0 = 0, A_1^1 = -\tfrac{3}{2}, \quad A_2^1 = 0,$$

$$A_3^1 = \tfrac{3}{4}, \quad A_6^1 = 1. \tag{21}$$

The required solution thus becomes

$$p = p_\infty + \left(-\tfrac{3}{2} r^{-3}\right) U x_3,$$

$$u_l = \left(\tfrac{3}{4} r^{-5} - \tfrac{3}{4} r^{-3}\right) U x_3 x_l + \left(-\tfrac{1}{4} r^{-3} - \tfrac{3}{4} r^{-1} + 1\right) U \delta_{l3}. \tag{22}$$

With the pressure and velocity components known, other quantities of interest, such as drag, etc., can be easily calculated. We also point out that solutions when the free stream velocity is of the form $\mathbf{u}_\infty = U_1 \mathbf{e}_1 + U_2 \mathbf{e}_2$ can be easily obtained by superposition of the solutions obtained in the above manner.

(ii) We next consider the linear shear flow past a solid spherical body. The boundary conditions in the cartesian-tensor form can be written as

$$p_\infty = 0, \; u_{l\infty} = a_{lj} x_j,$$

$$u_l = \tfrac{1}{2}(a_{lj} - a_{jl}) x_j \qquad \text{on} \quad r = 1. \tag{23}$$

The proper choices for the tensors a_{ij} and b_{ijk} in this case are

$$a_{ii} = 0, \quad b_{ijk} = 0 \qquad \text{for all} \quad i, j, k. \tag{24}$$

Consequently, the appropriate choice for the coefficients in (15) is

$$A_n^m = 0 \qquad \text{except when} \quad m = 5.$$

With this choice, the pressure and velocity vector in (6) and (9), become, respectively,

$$p = \left(A_1^5 r^{-5} + A_2^5\right) a_{ij} x_i x_j,$$

$$u_l = \left(\tfrac{1}{2} A_1^5 r^{-5} - \tfrac{2}{21} A_2^5 + A_3^5 r^{-7}\right) a_{ij} x_i x_j x_l + \left(\tfrac{5}{42} A_2^5 r^2 - \tfrac{1}{5} A_3^5 r^{-5} + A_5^5 r^{-3} + A_6^5\right) a_{lj} x_j$$

$$+ \left(\tfrac{5}{42} A_2^5 r^2 - \tfrac{1}{5} A_3^5 r^{-5} - A_5^5 r^{-3} + A_6^5\right) a_{jl} x_j. \tag{25}$$

The solution (25) will satisfy all the boundary conditions (23) if

$$A_1^5 = 5, \quad A_2^5 = 0 = A_5^5 = A_6^5, \quad A_3^5 = \tfrac{5}{2}, \quad A_6^5 = 1. \tag{26}$$

Equations (25) and (26) thus give the required solution as

$$p = -5r^{-5} a_{ij} x_i x_j,$$

$$u_l = \tfrac{5}{2}(r^{-7} - r^{-5}) a_{ij} x_i x_j x_l + \left(-\tfrac{1}{2}r^{-5} + 1\right) a_{lj} x_j + \left(-\tfrac{1}{2}r^{-5}\right) a_{jl} x_j. \tag{27}$$

We again point out that solutions when the shear flow is in another direction can be obtained in the above manner. One could also obtain the solution when the shear flow is acting simultaneously in different directions.

(iii) We next consider a quadratic or a paraboloidal flow past a sphere which is centred at the origin. We point out that the solution for an off-centred quadratic flow is equivalent to the centred flow superimposed on a uniform flow plus a shear flow. The boundary conditions for the quadratic flow are

$$p_\infty = 4kx_3, \quad \mathbf{u}_\infty = k(x^2 + y^2)\mathbf{e}_3, \quad \mathbf{u} = 0 \qquad \text{on} \quad r = 1. \tag{28}$$

where k is a constant. These boundary conditions can be written in suitable cartesian-tensor forms as

$$p_\infty = 2b_{imm} x_i, \quad u_{l\infty} = b_{ljk} x_j x_k, \quad u_l = 0 \qquad \text{on} \quad r = 1. \tag{29}$$

The appropriate choices for the tensors a_{ij} and b_{ijk} are found to be

$$a_{ij} = 0, \quad b_{311} = b_{322} = k, \quad b_{ijk} = 0 \qquad \text{otherwise}. \tag{30}$$

From (29), the natural choice for the coefficients in (15) is

$$A_n^m = 0 \quad \text{except when} \quad m = 2, 6. \tag{31}$$

Substitution of (31) in (15), and thence in (6) and (9), yields

$$\begin{aligned} p = {} & \left[A_1^2 r^{-3} + A_2^2 + 2(A_5^6 + A_7^6 + A_9^6) r^{-5}\right] b_{imm} x_i \\ & + \left[A_2^6 - 10(A_5^6 + A_7^6 + A_9^6) r^{-7}\right] b_{ijk} x_i x_j x_k, \end{aligned} \tag{32}$$

$$\begin{aligned} u_l = {} & \left[\tfrac{1}{2}A_1^2 r^{-3} + A_3^2 r^{-5} - \tfrac{1}{7}A_3^6 r^{-7} - \tfrac{1}{10}A_2^2 - 2A_6^6 + 3(A_8^6 + A_{10}^6) - \tfrac{1}{60}A_2^6 r^2\right] b_{imm} x_i x_l \\ & + \Big[\tfrac{1}{2}A_1^2 r^{-1} + A_6^2 + \tfrac{1}{35}A_3^6 r^{-5} + \tfrac{1}{3}(A_7^6 + A_9^6 - A_3^2) r^{-3} \\ & \quad + \tfrac{1}{20}\{4(A_2^2 - 2A_6^6) + 2(A_8^6 + A_{10}^6)\} r^2 - \tfrac{1}{72}A_2^6 r^4\Big] b_{lmm} \\ & + \left[A_3^6 r^{-9} - 5(A_5^6 + A_7^6 + A_9^6) r^{-7} - \tfrac{1}{12}A_2^6\right] b_{ijk} x_i x_j x_k x_l \end{aligned}$$

$$+\left[-\tfrac{1}{7}A_3^6 r^{-7}+A_5^6 r^{-5}+A_6^7+\tfrac{1}{12}A_2^6 r^2\right]b_{ljk}x_jx_k$$

$$+\left[-\tfrac{1}{7}A_3^6 r^{-7}+A_5^6 r^{-5}+A_6^6+\tfrac{1}{12}A_2^6 r^2\right]b_{jlk}x_jx_k$$

$$+\left[-\tfrac{1}{7}A_3^6 r^{-7}+A_9^6 r^{-5}+A_{10}^6+\tfrac{1}{12}A_2^6 r^2\right]b_{jkl}x_jx_k. \tag{33}$$

The general solutions (32) and (33) will satisfy the boundary conditions (29) provided

$$A_1^2=-\tfrac{1}{2},\quad A_2^2=2,\quad A_3^2=\tfrac{3}{8},\quad A_6^2=0,\quad A_3^6=\tfrac{35}{8},\quad A_5^6=-\tfrac{3}{8},$$

$$A_6^6=1,\quad A_7^6=\tfrac{5}{8},\quad A_8^6=-1,\quad A_0^6=\tfrac{5}{8},\quad A_{10}^6=1. \tag{34}$$

Hence the required solution becomes

$$p=\left(-\tfrac{1}{2}r^{-3}+\tfrac{7}{4}r^{-5}+2\right)b_{imm}x_i+\left(-\tfrac{35}{4}r^{-7}\right)b_{ijk}x_ix_jx_k, \tag{35}$$

$$u_l=\left(-\tfrac{1}{4}r^{-3}+\tfrac{3}{8}r^{-5}-\tfrac{5}{8}r^{-7}\right)b_{imm}x_ix_l+\left(\tfrac{1}{4}r^{-1}+\tfrac{1}{8}r^{-5}+\tfrac{7}{24}r^{-3}\right)b_{lmm}$$

$$+\left(\tfrac{35}{8}r^{-9}-\tfrac{35}{8}r^{-7}\right)b_{ijk}x_ix_jx_kx_l+\left(-\tfrac{5}{8}r^{-7}-\tfrac{3}{8}r^{-5}+1\right)b_{ljk}x_jx_k$$

$$+\left(-\tfrac{5}{8}r^{-7}+\tfrac{5}{8}r^{-5}-1\right)b_{jlk}x_jx_k+\left(-\tfrac{5}{8}r^{-7}-\tfrac{5}{8}r^{-5}+1\right)b_{jkl}x_jx_k. \tag{36}$$

We point out that the present solution agrees with Simha [4]. Moreover, the solution for quadratic flows in the other direction can be written down easily by the above approach.

(iv) As a final illustration we consider the slow uniform flow of an incompressible viscous fluid past a fluid sphere. This flow is similar to the one discussed in (i) except that now we have a fluid sphere and, therefore, the boundary conditions at the surface, assuming it remains spherical, will be different. Thus, if we distinguish the fluid inside and outside of the spherical drop by appending the superscripts i and e, respectively, the boundary conditions can then be written as

$$u_{l\infty}^{(e)}=\epsilon_{312}a_{21}=U\delta_{l3},$$

$u^{(i)}$ remains bounded at $r=0$,

$$u_l^{(e)}x_l=u_l^{(i)}x_l=0 \qquad \text{on} \quad r=1,$$

$$u_l^{(e)}-u_j^{(e)}x_jx_l=u_l^{(i)}-u_j^{(i)}x_jx_l \qquad \text{on} \quad r=1, \tag{37}$$

and

$$t_{ij}^{(e)}x_j-t_{kl}^{(e)}x_kx_lx_i=t_{ij}^{(i)}x_j-t_{kl}^{(i)}x_kx_lx_i \qquad \text{on} \quad r=1.$$

It should be pointed out that the third, fourth and fifth conditions in (37) represent the continuity of the normal component of the velocity, the tangential component of the velocity and the tangential stress on the surface, respectively, From (37) it appears that the appropriate choice in (15) is $A_n^m = 0$ for all except when $m = 1$, $a_{21} = U$, $a_{ij} = 0$ otherwise, and $b_{ijk} = 0$ for all i, j, k. Thus, we write

$$p^{(e)} = \left(A_1^1 r^{-3} + A_2^1\right) U x_3,$$

$$u_l^{(e)} = \left(A_3^1 r^{-5} + \tfrac{1}{2} A_1^1 r^{-3} - \tfrac{1}{10} A_2^1\right) U x_3 x_l + \left(-\tfrac{1}{3} A_3^1 r^{-3} + \tfrac{1}{2} A_1^1 r^{-1} + A_6^1 + \tfrac{1}{5} A_2^1 r^2\right) U \delta_{l3}, \tag{38}$$

with similar expressions for the fluid inside the drop. In particular, we replace A_n^m by B_q^p for the fluid inside the droplet and write

$$p^{(i)} = \left(B_1^1 r^{-3} + B_2^1\right) U x_3,$$

$$u_l^{(i)} = \left(B_3^1 r^{-5} + \tfrac{1}{2} B_1^1 r^{-3} - \tfrac{1}{10} B_2^1\right) U x_3 x_l + \left(-\tfrac{1}{3} B_3^1 r^{-3} + \tfrac{1}{2} B_1^1 r^{-1} + B_6^1 + \tfrac{1}{5} B_5^1 r^2\right) U \delta_{l3}. \tag{39}$$

In order that (38) and (39) satisfy the boundary conditions (37), it is necessary that

$$A_2^1 = 0, \quad A_6^1 = 1, \quad B_3^1 = B_1^1 = 0, \quad \tfrac{2}{3} A_3^1 + A_1^1 + 1 = 0,$$

$$B_6^1 + \tfrac{1}{10} B_2^1 = 0, \quad \tfrac{1}{3} A_3^1 - \tfrac{1}{2} A_1^1 - 1 = -B_6^1 - \tfrac{1}{5} B_2^1,$$

$$2\mu^{(e)} A_3^1 = \frac{3\mu^{(i)}}{10} B_2^1. \tag{40}$$

On solving the above system of equations we find

$$A_3^1 = \frac{3}{4(1+\sigma)}, \quad A_1^1 = -\frac{2\sigma + 3}{2(1+\sigma)}, \quad A_6^1 = 1, \quad A_2^1 = 0,$$

$$B_3^1 = 0, \quad B_1^1 = 0, \quad B_6^1 = \frac{\sigma}{2(1+\sigma)}, \quad B_2^1 = \frac{5\sigma}{1+\sigma} \tag{41}$$

where $\sigma = \mu^{(e)}/\mu^{(i)}$ is the ratio of the viscosities. On substituting these values in (38) and (30) we obtain the desired solution as

$$p^{(e)} = -\frac{2\sigma + 3}{2(1+\sigma)} r^{-3} U x_3,$$

$$u_l^{(e)} = \left[\frac{3}{4(1+\sigma)} r^{-5} - \frac{2\sigma + 3}{4(1+\sigma)} r^{-3}\right] U x_3 x_l + \left[-\frac{1}{4(1+\sigma)} r^{-3} - \frac{2\sigma + 3}{4(1+\sigma)} r^{-1} + 1\right] U \delta_{l3}, \tag{42}$$

$$p^{(i)} = \frac{5\sigma}{1+\sigma} U x_3,$$

$$u_l^{(i)} = -\frac{\sigma}{2(l+\sigma)} U x_3 x_l + \left[-\frac{\sigma}{2(l+\sigma)} + \frac{\sigma}{l+\sigma} r^2 \right] U \delta_{l3}. \tag{43}$$

Acknowledgements

The work reported in this paper has been supported by Grant No. A7728 of N.S.E.R.C. of Canada.

References

[1] J. Happel and H. Brenner, *Low Reynolds number hydrodynamics*, Chapter 3, Noordhoff Int. Publ., Leyden (1973).

[2] H. Lamb, *Hydrodynamics*, 6th edn., London: Cambridge University Press, New York: Dover (1945).

[3] H.A. Lorentz, A general theorem concerning the motion of a viscous fluid and a few consequences derived from it, *Versl. Kon. Akad. Wet. Amst.* 5 (1897) 168–175.

[4] R. Simha, Untersuchungen über die Viskosität von Suspensionen und Lösungen, *Kolloid Z.* 76 (1936) 16–19.